ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 2

MATH00040

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2. Complex Numbers

2.1. **Introduction to Complex Numbers.**

The first thing that it is important to realise is that complex numbers are not particularly complex, the next thing is that real numbers are not any more 'real' than imaginary numbers. These are just words that mathematicians have given them, so there is nothing to be worried about!

To give an idea of where complex numbers lie in the general scheme of things, let us have a look at various types of numbers. First and foremost there are the *natural numbers*, i.e., the counting numbers. The set $\{1, 2, 3, \ldots\}$ of all natural numbers is denoted $\mathbb{N}$ (note that some texts take $\mathbb{N} = \{0, 1, 2, \ldots\}$). These numbers are fine for counting things or for solving equations such as $x+1 = 2$, but a problem arises if we want to solve an equation such as $x + 2 = 1$, for example. In order to remedy this and other problems, negative whole numbers and the number zero were introduced. The natural numbers together with these are called the *integers* and are denoted $\mathbb{Z}$. So we have $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Again the integers are fine for doing some sorts of mathematics but problems now arise if we want to solve equations such as $2x = 1$, for example. In order to remedy this, yet another class of numbers had to be introduced. These are numbers of the form $\dfrac{x}{y}$, with $x, y \in \mathbb{Z}$ and $y \neq 0$, the so called *rational* numbers, which are denoted $\mathbb{Q}$. So we can write $\mathbb{Q} = \left\{\dfrac{x}{y} : x, y \in \mathbb{Z}, y \neq 0\right\}$. As you might be expecting, the rational numbers are fine for some things but not for others. For example, they are no good if we want to solve the equation $x^2 = 2$. Accordingly the so called *real* numbers were introduced. These are denoted $\mathbb{R}$ but I will not define real numbers in this course. There is a very good reason for this: you may think that you are familiar with real numbers and that you know what they are, but in fact they are extremely complicated animals and it could easily take a whole third year course to introduce them properly!

We then come to the *complex* numbers. As the real numbers were introduced to remedy a defect in the rational numbers, complex numbers were introduced to remedy a defect in the real numbers, i.e., there are no real solutions of equations such as $x^2+1=0$. Let us start by defining complex numbers and then see how we can add, multiply and divide one complex number by another. In order to define complex numbers we will need the symbol i, which has the property that when you square it you get the number -1. It is important not to worry too much about what this means, it is much better to take it as a definition. It is really quite similar to the way $\sqrt{2}$ was introduced, we *defined* $\sqrt{2}$ to be the number that when squared gives the number 2.

Definition 2.1.1 (Complex number)**.** A *complex number* is a number of the form $a+bi$ where a and b are real numbers and i has the property that $i^2=-1$.

Remark 2.1.2. There are other forms a complex number can be written in, but when it is in the form $a+bi$, then we say it is in *Cartesian form*.

Given a complex number $a+bi$ we will sometimes want to refer to the a or the b, so we give them special names.

Definition 2.1.3 (Real and Imaginary parts)**.** Given a complex number $a+bi$, then the *real* part of $a+bi$ is a and the *imaginary* part of $a+bi$ is b. In symbols we say $\mathrm{Re}(a+bi)=a$ and $\mathrm{Im}(a+bi)=b$.

Warning 2.1.4. Note that the imaginary part of $a+bi$ is defined to be b, not bi. That is the imaginary part of a complex number is a real number.

It can often be helpful to use a graphical representation of complex numbers, the so called *Argand diagram*. This identifies the complex number $a+bi$ with the point (a,b) in $\mathbb{R}^2$. A selection of complex numbers is represented in Figure 1.

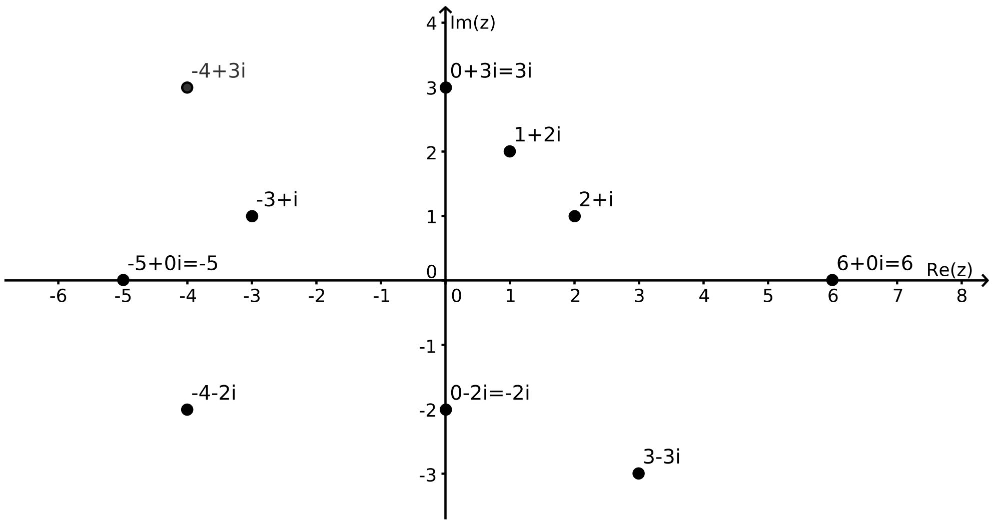

FIGURE 1. An Argand diagram with complex numbers as points.

Remark 2.1.5. Note that when we are dealing with an unknown complex number, we usually denote it by z rather than the x we would use with a real number. Instead of y, we usually use w.

Sometimes, instead of representing complex numbers as points, we represent them as vectors from the origin. Some examples of this are shown in Figure 2.

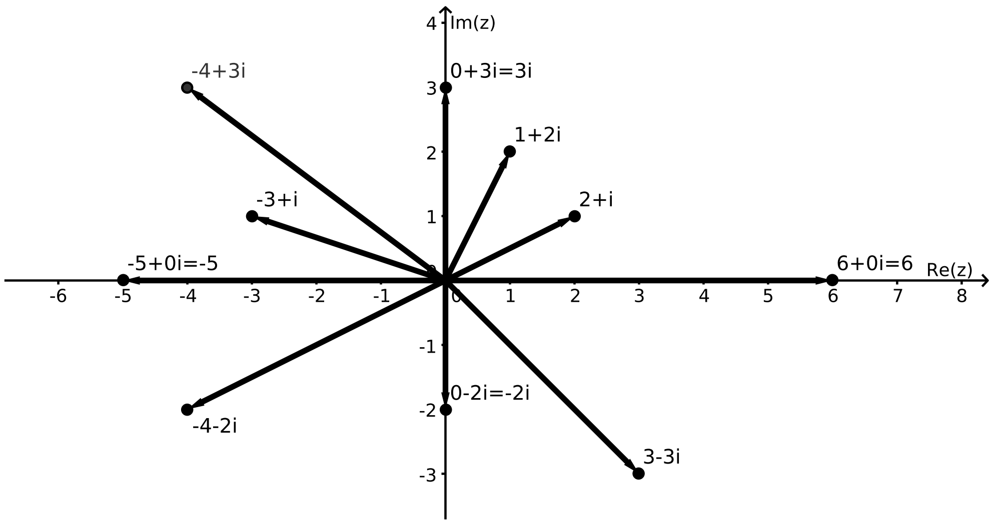

FIGURE 2. An Argand diagram with complex numbers as vectors.

We define the absolute value (sometimes called the modulus) of a complex number as its distance from the origin in the Argand diagram.

Definition 2.1.6 (Absolute value (modulus) of a complex number)**.** The *absolute value* (or *modulus*) of a complex number is given by $|a + bi| = \sqrt{a^2 + b^2}$.

2.2. **Arithmetic of Complex Numbers.**

Now that we have an idea as to what complex numbers are, we will have to define how to add, subtract, multiply and divide them. Addition, subtraction and multiplication are all straightforward and are performed as follows.

Definition 2.2.1 (Addition of complex numbers)**.** Given two complex numbers $a + bi$ and $c + di$, then their sum is defined to be

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

That is to add two complex numbers, we add their real parts and add their imaginary parts.

Example 2.2.2. The following are some concrete examples.

(1) $(1 + 2i) + (2 + 3i) = (1 + 2) + (2 + 3)i = 3 + 5i$.
(2) $(2 - i) + (-1 + 5i) = (2 - 1) + (-1 + 5)i = 1 + 4i$.
(3) $(-1 - 2i) + (1 - 2i) = (-1 + 1) + (-2 - 2)i = 0 - 4i = -4i$.

(4) $(3+3i)+(2-3i)=(3+2)+(3-3)i=5+0i=5$.

Subtraction of complex numbers is similar.

Definition 2.2.3 (Subtraction of complex numbers)**.** Given two complex numbers $a+bi$ and $c+di$, then $a+bi$ minus $c+di$ is defined to be

$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

That is to subtract one complex number from another, we subtract their real parts and subtract their imaginary parts.

Example 2.2.4. Here are some concrete examples.

(1) $(1+2i)-(2+3i)=(1-2)+(2-3)i=-1-i$.
(2) $(2-i)-(-1+5i)=(2-(-1))+(-1-5)i=3-6i$.
(3) $(-1-2i)-(-1+2i)=(-1-(-1))+(-2-2)i=0-4i=-4i$.
(4) $(3+3i)-(2+3i)=(3-2)+(3-3)i=1+0i=1$.

Remark 2.2.5. If we regard complex numbers as vectors in $\mathbb{R}^2$, then addition and subtraction of complex numbers may be regarded as addition and subtraction of vectors in the usual manner.

Multiplication of complex numbers is accomplished in a similar manner to multiplying expressions such as $(a+bx)$ and $(c+dx)$, where at the end we just have to let $x=i$ and $x^2=-1$. Remember that

$$(a+bx)(c+dx)=ac+adx+bcx+bdx^2=ac+(ad+bc)x+bdx^2$$

and this yields the following definition.

Definition 2.2.6 (Multiplication of complex numbers)**.** Given two complex numbers $a+bi$ and $c+di$, then $a+bi$ multiplied by $c+di$ is defined to be

$$(a+bi)(c+di)=ac+adi+bci+bdi^2=ac+(ad+bc)i-bd=(ac-bd)+(ad+bc)i.$$

That is to multiply one complex number by another, we multiply out the brackets in the usual manner and let $i^2=-1$ at the end.

Example 2.2.7. Again here are some concrete examples.

(1) $(1+2i)(2+3i)=((1)(2)-(2)(3))+((1)(3)+(2)(2))i=-4+7i$.
(2) $(2-i)(-1+5i)=((2)(-1)-(-1)(5))+((2)(5)+(-1)(-1))i=3+11i$.
(3) $(3-2i)(3+2i)=((3)(3)-(-2)(2))+((3)(2)+(-2)(3))i=13+0i=13$.
(4) $(3-2i)(-2+3i)=((3)(-2)-(-2)(3))+((3)(3)+(-2)(-2))i=0+13i=13i$.

Remark 2.2.8. Note that if we add, subtract or multiply complex numbers, we can end up with a real number or with a number that is purely imaginary or a mixture of both.
As with real numbers (but in contrast to matrices), if we have $zw=0$ then at least one of z or w must be zero.

We still have to deal with division of one complex number by another. While not particularly difficult, we do need to follow a procedure and for this we need the following definition.

Definition 2.2.9 (Complex conjugate)**.** Given a complex number $z = a + bi$, then the *complex conjugate* of z is defined to be $\overline{z} = a - bi$. That is to find the complex conjugate of a complex conjugate, we simply change the sign of the imaginary part. For example $\overline{2 + 3i} = 2 - 3i$ and $\overline{-2 - 5i} = -2 + 5i$.

Using this definition, it is now easy to explain how to divide $a + bi$ by $c + di$. The procedure is to multiply the top and bottom of $\dfrac{a + bi}{c + di}$ by the complex conjugate of $c + di$. Thus

$$\begin{aligned}
\frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\
&= \frac{(ac+db)+(bc-ad)i}{(c^2+d^2)+(cd-cd)i} \\
&= \frac{(ac+db)+(bc-ad)i}{c^2+d^2} \\
&= \frac{ac+db}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i
\end{aligned}$$

Remark 2.2.10. Rather than remember this equation, it is much easier to just remember that you have to multiply the top and the bottom of the quotient by the complex conjugate of the bottom, and then multiply out using the usual rules combined with $i^2 = -1$.

Example 2.2.11. Again a couple of examples may be useful.

(1)
$$\frac{2+3i}{3-4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)} = \frac{(6-12)+(8+9)i}{3^2+4^2} = -\frac{6}{25} + \frac{17}{25} i.$$

(2)
$$\frac{1-2i}{i} = \frac{(1-2i)(-i)}{(i)(-i)} = \frac{-2-i}{1} = -2 - i.$$

Please take special note of this second example, since it is one that often causes problems. If we let $a = 0$ in $\overline{a + bi} = a - bi$ we obtain $\overline{bi} = -bi$, so for example, the complex conjugate of i is $-i$.

2.3. **Polar form and DeMoivre's Formula**.

While it is fine to represent complex numbers as $a + bi$ for addition, subtraction, multiplication and division, there is a more useful representation if we want to take powers of complex numbers. This is the so called *polar form* and the idea is to define the position of a non-zero complex number z on the Argand diagram by giving its distance from the origin and the angle the vector from the origin to z makes with the positive real axis (where anti-clockwise angles count as positive and clockwise angles count as negative). How this works is shown in Figure 3.

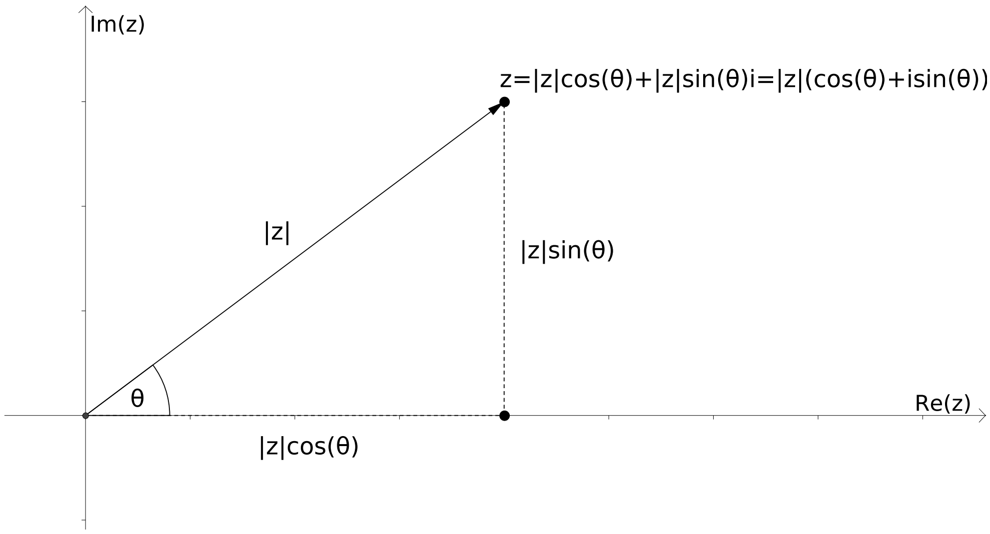

FIGURE 3. The polar form of a complex number.

Definition 2.3.1 (Polar form of a complex number)**.** A complex number is said to be in *polar form* if it is expressed in the form $r(\cos(\theta)+i\sin(\theta))$, where $r,\theta\in\mathbb{R}$ with $r>0$. In this representation, r is the distance the complex number lies from the origin (i.e., it is the modulus of the complex number) and θ is the angle the vector from the origin to the complex number makes with the positive real axis. This angle is called the *argument* of the complex number.

Remark 2.3.2. Note that the argument of a complex number is not unique: if n is an integer then $\cos(\theta+2n\pi)=\cos(\theta)$ and $\sin(\theta+2n\pi)=\sin(\theta)$, so it follows that if θ is an argument of a complex number then so is $\theta+2n\pi$.

Often we will want to convert a complex number in Cartesian form to polar form and vice-versa. If a complex number is in polar form $r(\cos(\theta)+i\sin(\theta))$ then it is easy to convert it to Cartesian form since we simply have to calculate $r\cos(\theta)$ and $r\sin(\theta)$. Going in the opposite direction is a bit more tricky though.

Say we want to convert a complex number $z=a+bi$ into polar form. There are four possibilities depending on whether a and b are positive or negative. We will first consider the easiest case when a and b are positive.

In this case, we can see from Figure 4 that $\tan(\theta)=\dfrac{b}{a}$. Hence $\theta=\tan^{-1}\left(\dfrac{b}{a}\right)$. Since $r=\sqrt{a^2+b^2}$, we have now expressed z in the form $r(\cos(\theta)+i\sin(\theta))$.

Remark 2.3.3. Note that this formula doesn't work if $a=0$ but this case is easy to deal with. Since $b>0$, the number z lies on the positive imaginary axis and so $\theta=\dfrac{\pi}{2}$. Thus in this case $z=b\left(\cos\left(\dfrac{\pi}{2}\right)+i\sin\left(\dfrac{\pi}{2}\right)\right)$.

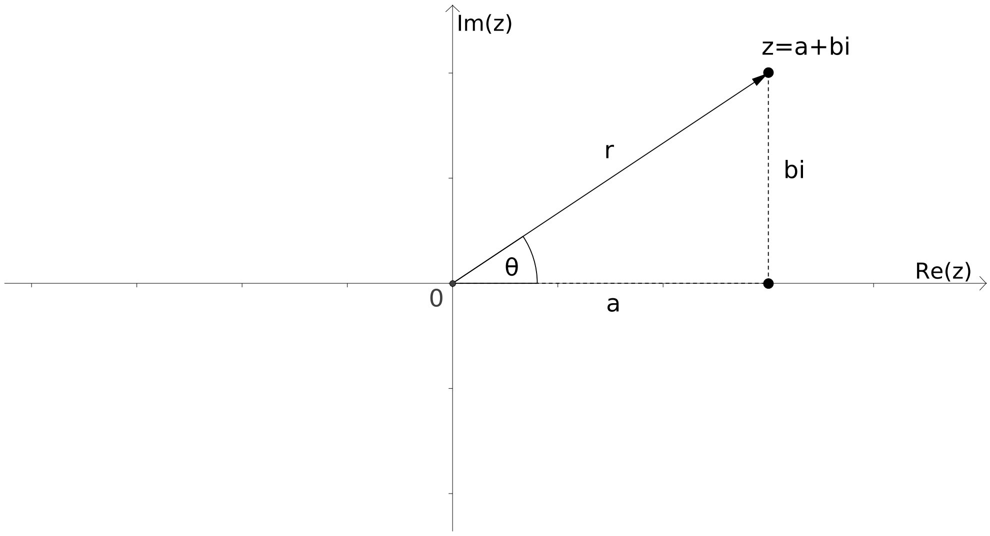

FIGURE 4. Converting a complex number from Cartesian to polar form (a and b positive.

If one or both of a and b are negative then things are a bit more complicated. In this case we first have to calculate another angle ϕ and then find what θ corresponds to this ϕ.

Let us next consider the case where a is positive and b is negative.

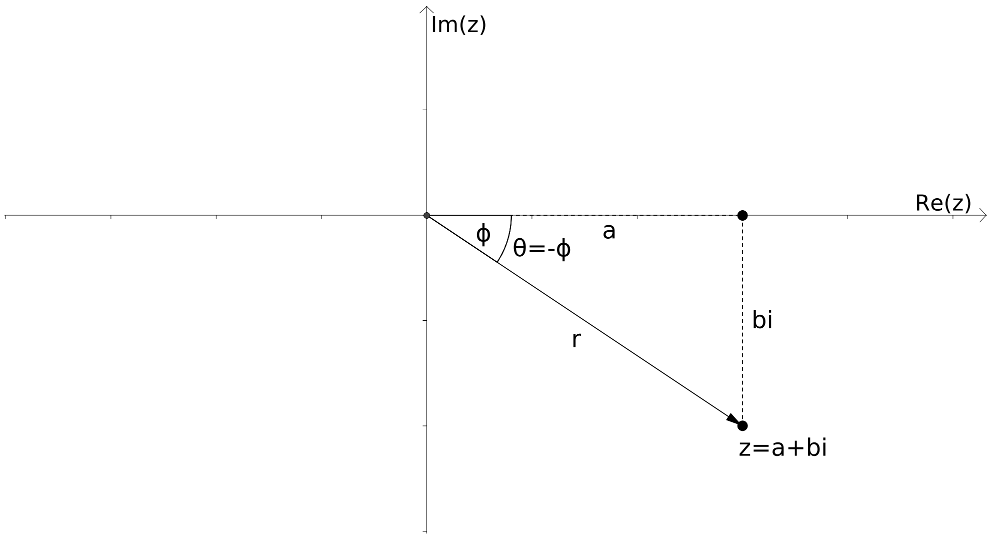

FIGURE 5. Converting a complex number from Cartesian to polar form (a positive and b negative.

The idea is to first calculate ϕ (see Figure 5) which is just an angle without regard of direction and then express θ (which does take direction into account) In terms of ϕ. If we look at Figure 5, we can see that $\tan(\phi) = \left|\frac{b}{a}\right|$. Note that we have to take the absolute value of $\frac{b}{a}$ for otherwise we would get a negative angle for ϕ and that is not what we want. Hence $\phi = \tan^{-1}\left(\left|\frac{b}{a}\right|\right)$. If we now remember that clockwise angles of θ count as negative, we see that $\theta = -\phi = -\tan^{-1}\left(\left|\frac{b}{a}\right|\right)$. Since we always have $r = \sqrt{a^2+b^2}$, we have now expressed z in polar form.

Remark 2.3.4. Again note that this formula doesn't work if $a = 0$ but this case is also easy to deal with. Since $b < 0$, the number z lies on the negative imaginary axis and so $\theta = -\frac{\pi}{2}$. Thus in this case $z = |b|\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$, where we have to take the absolute value of b to end up with a positive number.

Next let us deal with the case where a is negative and b is positive.

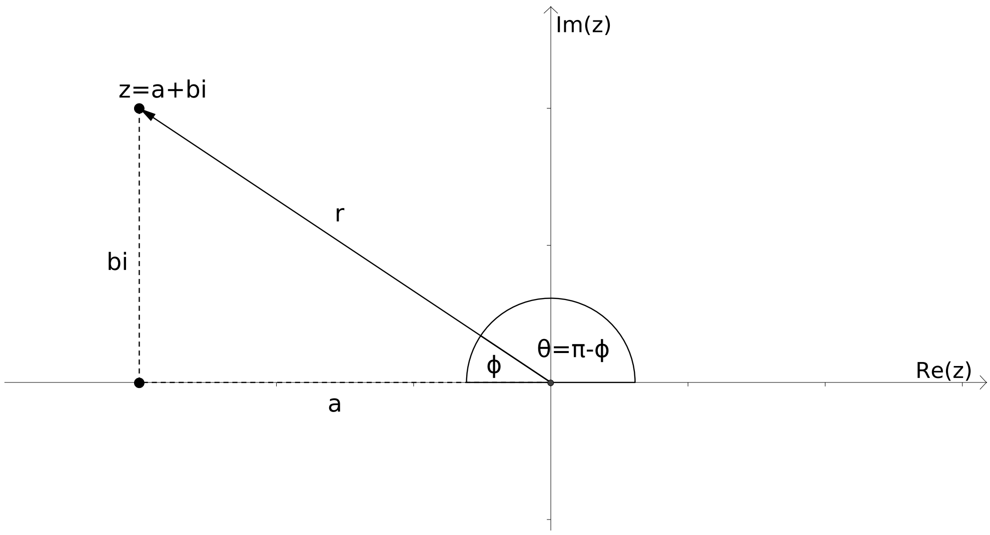

FIGURE 6. Converting a complex number from Cartesian to polar form (a negative and b positive).

The calculation for r and ϕ here is the same but we have a different final expression for θ. In this case $\theta = \pi - \phi = \pi - \tan^{-1}\left(\left|\frac{b}{a}\right|\right)$.

The final case is where both a and b are negative and this case is shown in Figure 7. Again the calculation for r and θ is the same but this time θ is given by

$$\theta = \phi - \pi = \tan^{-1}\left(\left|\frac{b}{a}\right|\right) - \pi.$$

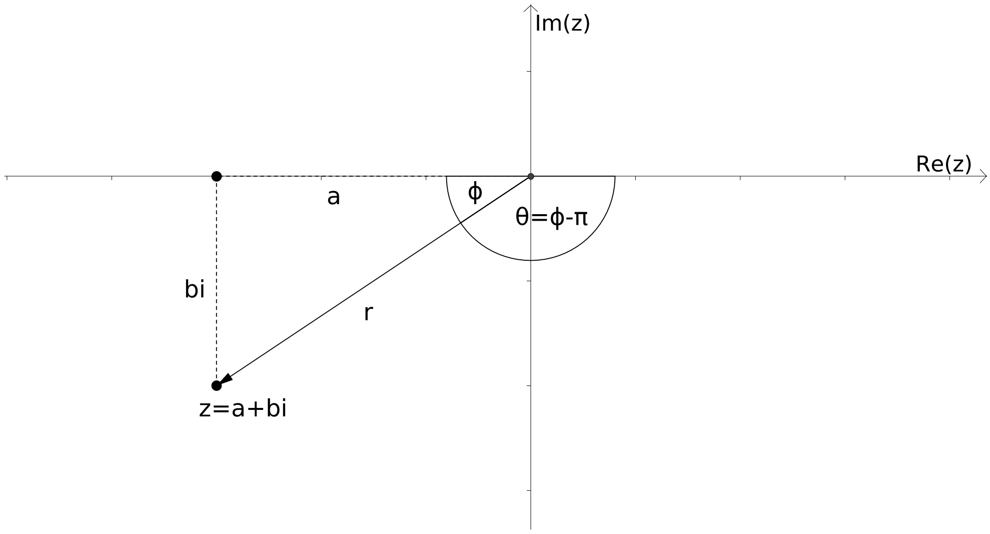

FIGURE 7. Converting a complex number from Cartesian to polar form (a and b negative).

Example 2.3.5. Let us do an example of each type to see how this works in practice.

(1) Let $z = 1 + i$, so that $a = 1$ and $b = 1$ (i.e., a and b are both positive). So we are in the situation shown in Figure 4. Hence $\tan(\theta) = \frac{1}{1} = 1$ and
$$\theta = \tan^{-1}(1) = \frac{\pi}{4}.$$
Next $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Thus $z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$ in polar form.

(2) Let $z = 1 - \sqrt{3}i$, so that $a = 1$ and $b = -\sqrt{3}$ (i.e., a is positive and b is negative). So we are in the situation shown in Figure 5. In this case we have to calculate ϕ first. Now
$$\tan(\phi) = \left|\frac{-\sqrt{3}}{1}\right| = \sqrt{3}$$
and $\phi = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$. Thus $\theta = -\phi = -\frac{\pi}{3}$. Next
$$r = |z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2.$$
Thus $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$ in polar form. Note that we leave the minus sign in the brackets and don't try to simplify using $\cos(-\theta) = \cos(\theta)$ or $\sin(-\theta) = -\sin(\theta)$.

(3) Let $z = -\sqrt{3} + i$, so that $a = -\sqrt{3}$ and $b = 1$ (i.e., a is negative and b is positive). So we are in the situation shown in Figure 6. Again we have to

calculate ϕ first. Now

$$\tan(\phi) = \left|\frac{1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$$

and $\phi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. Thus $\theta = \pi - \phi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Next

$$r = |z| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2.$$

Thus $z = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$ in polar form.

(4) Let $z = -2-2i$, so that $a = -2$ and $b = -2$ (i.e., a and b are both negative). So we are in the situation shown in Figure 7. Again we have to calculate ϕ first. Now

$$\tan(\phi) = \left|\frac{-2}{-2}\right| = 1$$

and $\phi = \tan^{-1}(1) = \frac{\pi}{4}$. Thus $\theta = \phi - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$. Next

$$r = |z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

Thus $z = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$ in polar form.

Warning 2.3.6. I have used numbers in Example 2.3.5 for which the angles come out in exact multiples of π. In general we would have to use a calculator to find ϕ and θ.

It may seem that it is a lot of work to convert a complex number from Cartesian to polar form but we will shortly come to the reason it is all worthwhile. However before that we will note that it is a bit easier to multiply complex numbers in polar form than it is to multiply them when they are in Cartesian form.

Definition 2.3.7 (Multiplying complex numbers in polar form)**.** Given two complex numbers in polar form $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

That is to multiply two complex numbers in polar form, we multiply their moduli and add their arguments.

Now for the main reason we went to the trouble of expressing complex numbers in polar form.

Theorem 2.3.8 (DeMoivre's Formula)**.** *Let* $z = \cos(\theta) + i\sin(\theta)$ *and let* $n \in \mathbb{Z}$. *Then*

$$z^n = \cos(n\theta) + i\sin(n\theta).$$

That is to raise a complex number of the form $\cos(\theta) + i\sin(\theta)$ *to an integer power* n*, we multiply the argument by* n*.*

Using Theorem 2.3.8 and the rule of indices $(rz)^n = r^n z^n$, this allows us to raise any complex number in polar form to any integer power.

Corollary 2.3.9. *Let $z = r(\cos(\theta) + i\sin(\theta))$ and let $n \in \mathbb{Z}$. Then*

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

That is to raise a complex number in polar form to an integer power n, we raise the modulus to the power n and multiply the argument by n.

Remark 2.3.10.
(1) As usual z^{-n} is defined to be $\dfrac{1}{z^n}$.
(2) This is a very powerful result, since it allows us to easily raise complex numbers to huge powers which would be almost impossible otherwise. To see what I mean, think about how you would find $(1+3i)^{1000}$.
(3) There is a generalization of DeMoivre's Formula in which we can take n to be any real number but we won't study it in this course since it adds some extra complications.

Example 2.3.11. The main work in finding a power of a complex number is in converting it to polar form, so we will use the solutions in Example 2.3.5 to calculate some powers.

(1) Find $(1+i)^{50}$. Using Corollary 2.3.9 we have

$$\begin{aligned}(1+i)^{50} &= \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right)^{50}\\ &= \left(\sqrt{2}\right)^{50}\left(\cos\left(\frac{50\pi}{4}\right) + i\sin\left(\frac{50\pi}{4}\right)\right)\\ &= 2^{25}(0+1i)\\ &= 2^{25}i.\end{aligned}$$

(2) Find $(1-\sqrt{3}i)^{15}$. Using Corollary 2.3.9 we have

$$\begin{aligned}(1-\sqrt{3}i)^{15} &= \left(2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right)^{15}\\ &= 2^{15}\left(\cos\left(-5\pi\right) + i\sin\left(-5\pi\right)\right)\\ &= 2^{15}(-1+0i)\\ &= -2^{15}.\end{aligned}$$

(3) Find $(-\sqrt{3}+i)^{60}$. Using Corollary 2.3.9 we have

$$\begin{aligned}(-\sqrt{3}+i)^{60} &= \left(2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)\right)^{60}\\ &= 2^{60}\left(\cos\left(50\pi\right) + i\sin\left(50\pi\right)\right)\\ &= 2^{60}(1+0i)\\ &= 2^{60}.\end{aligned}$$

(4) Find $(-2-2i)^{47}$. Using Corollary 2.3.9 we have

$$\begin{aligned}(-2-2i)^{47} &= \left(2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)\right)^{47}\\ &= 2^{\frac{141}{2}}\left(\cos\left(-\frac{141\pi}{4}\right)+i\sin\left(-\frac{141\pi}{4}\right)\right)\\ &= 2^{\frac{141}{2}}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\\ &= 2^{70}(-1+i).\end{aligned}$$

Warning 2.3.12. As in Example 2.3.5, I have used numbers for which the final answers are exact. In general we would have to use a calculator to find these.

2.4. **Roots of Complex Numbers**.

In this section, we will look at how we can find solutions to equations of the form $z^n = a+bi$, where $a+bi$ is a given complex number, n is a natural number and we have to find z. If we go back to real numbers for a moment and consider the equation $x^2 = 4$, then we see that there are two solutions $x = -2$ and $x = +2$. Note that the power of x is two and there are two solutions. Now let us consider the equation $x^4 = 16$. While this equation also has two real solutions $x = -2$ and $x = +2$, it has four complex solutions ($x = 2i$ and $x = -2i$ as well as the real solutions). So the power of x is four and the equation has four complex solutions but only two real solutions. We will see below that if $n \in \mathbb{N}$, then the equation $z^n = a+bi$ always has n complex solutions, so in some sense complex equations are more predictable than real equations.

Remark 2.4.1. Instead of saying we are finding the solutions of the equation $z^n = a+bi$, we often say we are finding the nth *roots* of $a+bi$. Note this is the same as in the real case, where instead of saying we are solving the equation $x^n = a$, we say we are finding the nth roots of a.

The solutions to the equation $z^n = a+bi$ (i.e., the nth roots of $a+bi$) are found using the following procedure.

(1) Express $a+bi$ in polar form, that is, find $r > 0$ and $\theta \in \mathbb{R}$ such that $a+bi = r(\cos(\theta)+i\sin(\theta))$.
(2) The nth roots are then given by

$$z_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right) \qquad k = 0,1,\ldots,n-1.$$

That is there are n roots indexed by k.

Remark 2.4.2. If we plot the roots on an Argand diagram (see Figure 8 for an example of the fifth roots), we will see that they are all the same distance from the origin (i.e., they lie on a circle centred at the origin) and also they are evenly distributed around the origin.

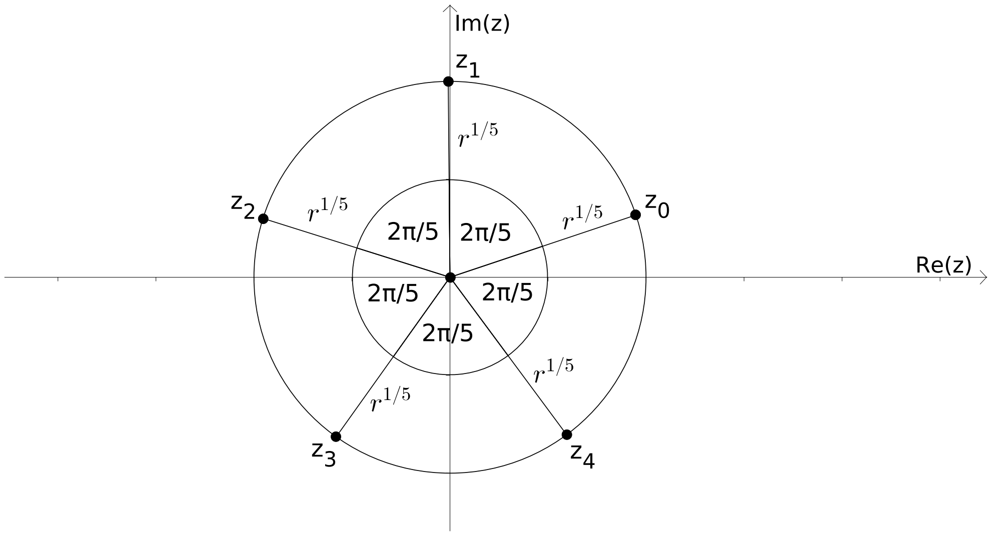

FIGURE 8. Fifth roots of a complex number.

Example 2.4.3. As is usual in Maths, a couple of examples will make things clearer.

(1) Find the fifth roots of $1+i$. Using Example 2.3.5.1,

$$1+i=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$$

in polar form. Hence the fifth roots of $1+i$ are given by

$$z_k=\left(\sqrt{2}\right)^{\frac{1}{5}}\left(\cos\left(\frac{\pi/4}{5}+\frac{2k\pi}{5}\right)+i\sin\left(\frac{\pi/4}{5}+\frac{2k\pi}{5}\right)\right)\qquad k=0,1,2,3,4.$$

That is the roots are

$$z_k=2^{\frac{1}{10}}\left(\cos\left(\frac{\pi}{20}+\frac{2k\pi}{5}\right)+i\sin\left(\frac{\pi}{20}+\frac{2k\pi}{5}\right)\right)\qquad k=0,1,2,3,4.$$

We can also write these out individually.

$$z_0=2^{\frac{1}{10}}\left(\cos\left(\frac{\pi}{20}\right)+i\sin\left(\frac{\pi}{20}\right)\right)$$
$$z_1=2^{\frac{1}{10}}\left(\cos\left(\frac{9\pi}{20}\right)+i\sin\left(\frac{9\pi}{20}\right)\right)$$
$$z_2=2^{\frac{1}{10}}\left(\cos\left(\frac{17\pi}{20}\right)+i\sin\left(\frac{17\pi}{20}\right)\right)$$
$$z_3=2^{\frac{1}{10}}\left(\cos\left(\frac{25\pi}{20}\right)+i\sin\left(\frac{25\pi}{20}\right)\right)$$
$$z_4=2^{\frac{1}{10}}\left(\cos\left(\frac{33\pi}{20}\right)+i\sin\left(\frac{33\pi}{20}\right)\right)$$

Note that if we add another $\dfrac{2\pi}{5}$ onto the argument of z_4 we get

$$2^{\frac{1}{10}}\left(\cos\left(\frac{41\pi}{20}\right)+i\sin\left(\frac{41\pi}{20}\right)\right)$$

and this is equal to z_0 since the only difference between them is that their arguments differ by 2π. Thus it is only possible to get five different roots in this case (and n different roots in the general case).
Of course it is also possible to use a calculator to find the roots in Cartesian form (I have given the roots to three decimal places).

$$\begin{aligned}
z_0 &\simeq 1.059+0.168i\\
z_1 &\simeq 0.168+1.059i\\
z_2 &\simeq -0.955+0.487i\\
z_3 &\simeq -0.758-0.758i\\
z_4 &\simeq 0.487-0.955i
\end{aligned}$$

(2) Find the third roots of $1-\sqrt{3}i$. Using Example 2.3.5.2,

$$1-\sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$$

in polar form. Hence the third roots of $1-\sqrt{3}i$ are given by

$$z_k = 2^{\frac{1}{3}}\left(\cos\left(\frac{-\pi/3}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{-\pi/3}{3}+\frac{2k\pi}{3}\right)\right)\qquad k=0,1,2.$$

That is the roots are

$$z_k = 2^{\frac{1}{3}}\left(\cos\left(-\frac{\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(-\frac{\pi}{9}+\frac{2k\pi}{3}\right)\right)\qquad k=0,1,2.$$

We can also write these out individually.

$$\begin{aligned}
z_0 &= 2^{\frac{1}{3}}\left(\cos\left(-\frac{\pi}{9}\right)+i\sin\left(-\frac{\pi}{9}\right)\right)\\
z_1 &= 2^{\frac{1}{3}}\left(\cos\left(\frac{5\pi}{9}\right)+i\sin\left(\frac{5\pi}{9}\right)\right)\\
z_2 &= 2^{\frac{1}{3}}\left(\cos\left(\frac{11\pi}{9}\right)+i\sin\left(\frac{11\pi}{9}\right)\right)
\end{aligned}$$

Again note that if we add another $\dfrac{2\pi}{3}$ onto the argument of z_2 we get

$$2^{\frac{1}{3}}\left(\cos\left(\frac{17\pi}{9}\right)+i\sin\left(\frac{17\pi}{9}\right)\right)$$

and this is equal to z_0 since the only difference between them is that their arguments differ by 2π. Thus it is only possible to get three different roots in this case.
In Cartesian form (to three decimal places),

$z_0 \simeq 1.184-0.431i, \quad z_1 \simeq -0.219+1.241i, \quad z_2 \simeq -0.965-0.810i.$